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On Gevrey regularity of globally C^∞ hypoelliptic operators[☆]

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Abstract

We prove Gevrey regularity for a class of operators defined on the torus \mathbb{T}^{m+n} , with real analytic coefficients in \mathbb{T}^m , and which are globally C^∞ hypoelliptic. Then, we apply this result to show global G^s hypoellipticity for a large class of sublaplacians that may include lower order terms.

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1. Introduction and preliminaries

Let $s \geq 1$. We say that a function $f(x) \in C^\infty(\mathbb{T}^N)$ is in the Gevrey class $G^s(\mathbb{T}^N)$ if there exists a constant $C > 0$ such that $|\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s$, for all $\alpha \in \mathbb{Z}_+^N$, $x \in \mathbb{T}^N$. In particular, $G^1(\mathbb{T}^N)$ is the space of all periodic analytic functions, denoted by $C^\omega(\mathbb{T}^N)$. One can prove that $u \in D'(\mathbb{T}^N)$ is in $G^s(\mathbb{T}^N)$ if and only if there exist

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positive constants ε and C such that

$$|\hat{u}(\xi)| \leq C e^{-\varepsilon|\xi|^{1/s}}, \quad \forall \xi \in \mathbb{Z}^N \setminus \{0\}.$$

A linear partial differential operator P defined on \mathbb{T}^N with coefficients in $C^\omega(\mathbb{T}^N)$ is said to be globally $G^s(C^\infty)$ hypoelliptic in \mathbb{T}^N if for any $u \in D'(\mathbb{T}^N)$ the condition $Pu \in G^s(\mathbb{T}^N)$ ($Pu \in C^\infty(\mathbb{T}^N)$) implies that $u \in G^s(\mathbb{T}^N)$ ($u \in C^\infty(\mathbb{T}^N)$). When $s = 1$ we say that P is globally analytic hypoelliptic in \mathbb{T}^N . Similar definitions can be given when we replace \mathbb{T}^N by a compact analytic manifold without boundary. If P is defined on an open set U of \mathbb{R}^N , then P is said to be locally $G^s(C^\infty)$ hypoelliptic if for any V open set of U and any $u \in D'(V)$ the condition $Pu \in G^s(V)$ ($Pu \in C^\infty(V)$) implies that $u \in G^s(V)$ ($u \in C^\infty(V)$). Note that local $G^s(C^\infty)$ hypoellipticity implies global $G^s(C^\infty)$ hypoellipticity. The opposite direction is not true in general. For example, in the C^∞ case, the operator $P = \partial_t + \alpha \partial_x$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a non-Liouville number, is globally C^∞ hypoelliptic in \mathbb{T}^2 but it is not locally C^∞ hypoelliptic in \mathbb{R}^2 , see Greenfield and Wallach [GW]. For more examples and results in the C^∞ case see, for example, Amano [A], Fujiwara and Omori [FO], Himonas [H1], Omori and Kobayashi [OK], and the references therein. For examples in the Gevrey framework we refer the reader to Section 2 of this paper.

Local and global analytic and C^∞ hypoellipticity has been studied by many authors, including Amano [A], Baouendi and Goulaouic [BG], Bell and Mohammed [BM], Bergamasco, Cordaro, and Malagutti [BCM], Bernardi, Bove and Tartakoff [BBT], Christ [C1–C4], Cordaro and Himonas [CH1, CH2], Derridj [D], Dickinson et al. [DGY], Fedii [F], Fujiwara and Omori [FO], Gramchev et al. [GPY2], Greenfield and Wallach [GW], Grigis and Sjöstrand [GS], Hanges and Himonas [HH1–HH3], Helffer [He], Himonas [H1], Himonas and Petronilho [HP1–HP3], Hörmander [Ho1], Kohn [K], Metivier [M], Oleinik and Radkevic [OR], Pham The Lai and Robert [PR], Rothschild and Stein [RS], Taira [T], Tartakoff [Ta1], Treves [Tr], and Chanillo et al. [CHL].

Also, local and global G^s hypoellipticity has been studied by many authors, e.g., Albanese et al. [ACR], Bove and Tartakoff [BT1, BT2], Christ [C5, C6], Gramchev et al. [GPY1], Himonas [H2], Rodino [R], and Tartakoff [Ta2]. We would like to point out that it is possible for an operator to be globally G^s hypoelliptic in \mathbb{T}^N without being globally C^∞ hypoelliptic. In fact, it follows from [GPY2] that if $1 \leq \sigma$ is fixed then there exists $a(t) \in C^\omega(\mathbb{T})$ such that the number $a_0 = \frac{1}{2\pi} \int_0^{2\pi} a(t) dt$ satisfies the condition: for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left| a_0 - \frac{p}{q} \right| \geq C_\varepsilon e^{-\varepsilon q^{1/\sigma}}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{N}.$$

It also follows from [GPY2] that the vector field

$$L = \partial_t + a(t)\partial_x$$

is globally G^s hypoelliptic in \mathbb{T}^2 if $1 \leq s \leq \sigma$, but L is not globally C^∞ hypoelliptic in \mathbb{T}^2 . Hence it is easy to see that the sublaplacian $P = L^2$ is globally G^s hypoelliptic in \mathbb{T}^2 if $1 \leq s \leq \sigma$, but P is not globally C^∞ hypoelliptic in \mathbb{T}^2 .

In this paper we are concerned with the following question: Suppose that P is a globally C^∞ hypoelliptic operator in \mathbb{T}^N , with real analytic coefficients. Is P globally G^s hypoelliptic in \mathbb{T}^N ?

We begin the discussion of our question by recalling the characterization of hypoellipticity for operators with constant coefficients. Let P be given by

$$P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}.$$

The symbol of P is given by

$$P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{Z}^N.$$

It follows from [GW] that P is globally C^∞ hypoelliptic in \mathbb{T}^N if and only if there exist positive constants L, M and C such that

$$|P(\xi)| \geq L|\xi|^{-M}, \quad |\xi| \geq C. \quad (1.1)$$

Also, it is easy to prove that P is globally G^s hypoelliptic in \mathbb{T}^N if and only if for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|P(\xi)| \geq e^{-\varepsilon|\xi|^{1/s}}, \quad |\xi| \geq C_\varepsilon. \quad (1.2)$$

Since (1.1) implies (1.2) we conclude that P has the following Gevrey regularity property:

(★) *If P is globally C^∞ hypoelliptic in \mathbb{T}^N then P is globally G^s hypoelliptic in \mathbb{T}^N .* Thus, when the operator has constant coefficients then our question has a positive answer.

In our main result we consider operators $P = P(t, D_t, D_x)$ defined on \mathbb{T}^{m+n} , with coefficients in $C^\omega(\mathbb{T}^m)$, that are globally C^∞ hypoelliptic in \mathbb{T}^{m+n} and we suppose that the kernel of P is contained in the Gevrey space $G^s(\mathbb{T}^{m+n})$. Then we prove Gevrey regularity of its solutions. Next, we use this result in order to present a large class of operators for which propriety (★) holds true. These operators are sums of squares of real vector fields including lower order terms. We show that Theorem 1.1 in Cordaro and Himonas [CH1] is a particular case of our main result. Also, we show that the operators that were proved by Christ [C6] and Bove and Tartakoff [BT2] to be not locally Gevrey hypoelliptic, are globally Gevrey hypoelliptic. These operators include the well known Baouendi and Goulaouic operator. We also show that the operators that

were proved in Hanges and Himonas [HH1] and in Christ [C1,C2] to be not locally analytic hypoelliptic, are globally Gevrey hypoelliptic.

We would like to point out that global C^∞ , analytic, and Gevrey hypoellipticity of sublaplacians is a difficult open problem in the theory of partial differential equations, since there is no known necessary and sufficient condition characterizing the hypoellipticity of a sublaplacian in its general form.

2. Main result and beginning of its proof

In order to state our main result we shall need the definition of the Gevrey wave front, which we recall below (see, for example, Hörmander [Ho2] or Rodino [R]).

Definition 2.1. If $X \subset \mathbb{R}^M$ and $u \in D'(X)$ we denote by $WF_s(u)$ the complement in $X \times (\mathbb{R}^M \setminus \{0\})$ of the set of (x_0, ξ_0) such that there is a neighborhood $U \subset X$ of x_0 , a conic neighborhood Γ of ξ_0 and a bounded sequence $u_N \in E'(X)$ which is equal to u in U and satisfies the condition

$$|\widehat{u_N}(\xi)| \leq C(CN/|\xi|^{1/s})^N, \quad N = 1, 2, \dots, \xi \in \Gamma$$

for some constant $C > 0$ independent of N .

For $s > 1$ this definition is equivalent to the following: $(x_0, \xi_0) \notin WF_s(u)$ if and only if there exists $\varphi \in G_0^s(X)$, $\varphi(x) = 1$ in a neighborhood of x_0 , and a conic neighborhood Γ of ξ_0 , such that for some $C, \varepsilon > 0$

$$|(\widehat{\varphi u})(\xi)| \leq C e^{-\varepsilon|\xi|^{1/s}} \quad \text{for all } \xi \in \Gamma.$$

We are now in position to present our main result.

Theorem 2.2. For $(t, x) \in \mathbb{T}^{m+n}$ let $P = P(t, D_t, D_x)$ be a linear partial differential operator with coefficients in $C^\omega(\mathbb{T}^m)$ and suppose that P is globally C^∞ hypoelliptic in \mathbb{T}^{m+n} . If $u \in D'(\mathbb{T}^{m+n})$, $Pu \in G^s(\mathbb{T}^{m+n})$, and $(t, x, \tau, 0) \notin WF_s(u)$, where $(t, x) \in \mathbb{T}^{m+n}$, $\tau \in \mathbb{R}^m \setminus \{0\}$, and if $\text{Ker } P \subset G^s(\mathbb{T}^{m+n})$, then $u \in G^s(\mathbb{T}^{m+n})$.

It follows from Theorem 2.2 (see Theorem 4.3) that operator in \mathbb{T}^3 given by

$$P = \partial_{t_1}^2 + [\partial_{t_2} - a(t_1)\partial_x]^2 \tag{2.1}$$

is globally G^s hypoelliptic in \mathbb{T}^3 if the function a is in $C^\omega(\mathbb{T})$, is real valued and not constant on \mathbb{T} . We would like to mention that operator (2.1) is globally analytic hypoelliptic (see [CH1]), but it is not in general locally analytic hypoelliptic. In Hanges and Himonas [HH1] it was proved that when $a(t_1) = t_1^{k-1}$, $k = 3, 5, 7, \dots$ then P is

not analytic hypoelliptic at 0. In [C1] Christ extended this result for all $k \geq 3$ and in [C2] he extended it for any analytic function $a(t_1)$ with $a(0) = a'(0) = 0$. The case $k = 3$ was first done by Helffer [He] and Pham The Lai and Robert [PR].

It also follows from Theorem 2.2 (see Theorem 4.3) that the following generalized Baouendi–Goulaouic operator

$$Q = \partial_t^2 + a^2(t)\partial_{x_1}^2 + b^2(t)\partial_{x_2}^2 \quad (2.2)$$

is globally G^s hypoelliptic in \mathbb{T}^3 , if a and b are real valued analytic functions which are not identically equal to zero. Operator (2.2) is globally analytic hypoelliptic (see [CH1]), but it is not in general locally Gevrey hypoelliptic. Christ [C6] has proved that, when $a(t) = t^{p-1}$ and $b(t) = t^{q-1}$, Q is Gevrey hypoelliptic for any $s \geq \frac{q}{p}$. On the other hand, Q is not Gevrey hypoelliptic in any class G^s with $s < \frac{q}{p}$. The results of Bove and Tartakoff [BT2] have refined those of Christ [C6]. More precisely, they have proved that Q is Gevrey hypoelliptic in any anisotropic Gevrey space G^{d_1, d_2, d_3} with $d_1 \geq 1 - \frac{1}{q} + \frac{1}{p}$, $d_2 \geq 1$ and $d_3 \geq \frac{q}{p}$. Q is not G^{d_1, d_2, d_3} hypoelliptic for any triple (d_1, d_2, d_3) such that $d_1 < 1 - \frac{1}{q} + \frac{1}{p}$, $d_2 < 1$ or $d_3 < \frac{q}{p}$. These operators include the well-known Baouendi–Goulaouic operator $\partial_t^2 + \partial_{x_1}^2 + t^2 \partial_{x_2}^2$.

In order to prove Theorem 2.2 we need some definitions and auxiliary results. For $\varphi \in C^\infty(\mathbb{T}^N)$, where $N = m + n$, and $j = 0, 1, 2, \dots$ we define

$$\|\varphi\|_j^2 = \sum_{|\alpha| \leq j} \|\partial^\alpha \varphi\|_0^2 = \sum_{|\alpha| \leq j} \int_{\mathbb{T}^N} |\partial^\alpha \varphi(x)|^2 dx.$$

We also define for $j = 0, 1, 2, \dots$ and $r = 0, 1, 2, \dots$

$$\|\varphi\|_{j,r} = \|P\varphi\|_j + \|\varphi\|_r, \quad \text{for } \varphi \in C^\infty(\mathbb{T}^N).$$

Lemma 2.3. *If P is globally C^∞ hypoelliptic in \mathbb{T}^N then $C^\infty(\mathbb{T}^N)$, with the topology defined by the sequence of norms $\{\|\cdot\|_{j,r}\}_j$, $j = 0, 1, 2, \dots$, is a complete metric space for any $r = 0, 1, 2, \dots$*

Proof. Let $r \in \mathbb{Z}_+$ be fixed. By using the sequence of norms $\{\|\cdot\|_{j,r}\}_j$, $j = 0, 1, 2, \dots$ we define the following metric on $C^\infty(\mathbb{T}^N)$:

$$d_1(\varphi, \psi) = \sum_{j=0}^{\infty} \frac{1}{2^j} \frac{\|\varphi - \psi\|_{j,r}}{1 + \|\varphi - \psi\|_{j,r}}, \quad \text{for } \varphi, \psi \in C^\infty(\mathbb{T}^N).$$

Thus $(C^\infty(\mathbb{T}^N), d_1)$ is a metric space. If φ_n is a Cauchy sequence in $(C^\infty(\mathbb{T}^N), d_1)$, then

$$\|\varphi_n - \varphi_m\|_{j,r} \rightarrow 0, \quad \forall j \in \mathbb{Z}_+. \quad (2.3)$$

Since

$$\|\varphi_n - \varphi_m\|_{j,r} = \|P\varphi_n - P\varphi_m\|_j + \|\varphi_n - \varphi_m\|_r,$$

then it follows from (2.3) that

$$\|P\varphi_n - P\varphi_m\|_j \rightarrow 0, \quad \forall j \in \mathbb{Z}_+ \quad (2.4)$$

and

$$\|\varphi_n - \varphi_m\|_r \rightarrow 0. \quad (2.5)$$

Since $H^j(\mathbb{T}^N)$ and $H^r(\mathbb{T}^N)$ are Hilbert spaces then it follows from (2.4) and (2.5) that there exist $\varphi \in H^r(\mathbb{T}^N)$, $\psi_j \in H^j(\mathbb{T}^N)$, $j \in \mathbb{Z}_+$ such that

$$\varphi_n \rightarrow \varphi \text{ in } H^r(\mathbb{T}^N) \quad (2.6)$$

and

$$P\varphi_n \rightarrow \psi_j \text{ in } H^j(\mathbb{T}^N), \quad j \in \mathbb{Z}_+. \quad (2.7)$$

Let $j < \ell$. It follows from (2.7) that $P\varphi_n \rightarrow \psi_\ell$ in $H^j(\mathbb{T}^N)$, since the inclusion $H^\ell(\mathbb{T}^N) \hookrightarrow H^j(\mathbb{T}^N)$ is continuous. Thanks again to (2.7) and the uniqueness of the limit in $H^j(\mathbb{T}^N)$ we obtain $\psi_j = \psi_\ell$ for $j, \ell \in \mathbb{Z}_+$. We set $\psi \doteq \psi_j = \psi_\ell$. It follows from (2.6) and (2.7), with $\psi_j = \psi$, that $\varphi_n \rightarrow \varphi$ in $D'(\mathbb{T}^N)$, $P\varphi_n \rightarrow \psi$ in $D'(\mathbb{T}^N)$, and $P\varphi_n \rightarrow P\varphi$ in $D'(\mathbb{T}^N)$. Thus, thanks to the uniqueness of the limit in $D'(\mathbb{T}^N)$ we obtain

$$P\varphi = \psi. \quad (2.8)$$

Note that

$$\psi \in C^\infty(\mathbb{T}^N) \quad (2.9)$$

since $\psi \in H^j(\mathbb{T}^N)$ for all $j \in \mathbb{Z}_+$. Conditions (2.8), (2.9) and the global C^∞ hypoellipticity of P imply that $\varphi \in C^\infty(\mathbb{T}^N)$. It follows from (2.8), (2.7) with $\psi_j = \psi$,

and (2.6) that

$$\begin{aligned}\|\varphi_n - \varphi\|_{j,r} &= \|P\varphi_n - P\varphi\|_j + \|\varphi_n - \varphi\|_r \\ &= \|P\varphi_n - \psi\|_j + \|\varphi_n - \varphi\|_r \rightarrow 0, \quad \text{as } n \rightarrow \infty.\end{aligned}\quad (2.10)$$

In the last equality we have replaced $\|P\varphi_n - P\varphi\|_j$ by $\|P\varphi_n - \psi\|_j$ since $P\varphi = \psi$ in $D'(\mathbb{T}^N)$. Thus $(C^\infty(\mathbb{T}^N), d_1)$ is a complete metric space. \square

Also, it is easy to prove the following lemma:

Lemma 2.4. *The space $C^\infty(\mathbb{T}^N)$ equipped with the metric d given by the sequence of norms $(\|\varphi\|_j)$, $j = 0, 1, 2, \dots$ is a complete metric space.*

Lemma 2.5. *If P is globally C^∞ hypoelliptic in \mathbb{T}^N then there exist $\ell \in \mathbb{Z}_+$ and $C > 0$ such that*

$$\|\varphi\|_1 \leq C(\|P\varphi\|_\ell + \|\varphi\|_0), \quad \forall \varphi \in C^\infty(\mathbb{T}^N). \quad (2.11)$$

Proof. Let $r \in \mathbb{Z}_+$ be fixed. For each $\ell \in \mathbb{Z}_+$ we define $k = \max\{m + \ell, r\}$ where m is the order of P . It is easy to see that there exists a constant $C > 0$ such that

$$\|\varphi\|_{\ell,r} = \|P\varphi\|_\ell + \|\varphi\|_r \leq C\|\varphi\|_k,$$

hence $Id : (C^\infty(\mathbb{T}^N), d) \rightarrow (C^\infty(\mathbb{T}^N), d_1)$ is continuous. We remind the reader that the global C^∞ hypoellipticity of P implies that $(C^\infty(\mathbb{T}^N), d_1)$ is a complete metric space. Since Id is an isomorphism it follows from the open mapping theorem that

$$Id^{-1} : (C^\infty(\mathbb{T}^N), d_1) \rightarrow (C^\infty(\mathbb{T}^N), d)$$

is continuous too. Hence for each $p \in \mathbb{Z}_+$ there exist $\ell \in \mathbb{Z}_+$ and $C > 0$ such that

$$\|\varphi\|_p \leq C(\|P\varphi\|_\ell + \|\varphi\|_r), \quad \text{for } \varphi \in C^\infty(\mathbb{T}^N). \quad (2.12)$$

By taking $p = 1$ and $r = 0$ the last inequality implies that there exist $\ell \in \mathbb{Z}_+$ and $C > 0$ such that

$$\|\varphi\|_1 \leq C(\|P\varphi\|_\ell + \|\varphi\|_0), \quad \text{for } \varphi \in C^\infty(\mathbb{T}^N).$$

The proof of Lemma 2.5 is complete. \square

Next, we show a key result needed for proving Theorem 2.1.

First, we note that $\text{Ker } P \subset C^\infty(\mathbb{T}^N)$ since P is globally C^∞ hypoelliptic in \mathbb{T}^N . Also, we denote by V the orthogonal space of $\text{Ker } P$ in $L^2(\mathbb{T}^N)$.

Theorem 2.6. *Suppose that P is globally C^∞ hypoelliptic in \mathbb{T}^N . Then there exist $\ell \in \mathbb{Z}_+$ and $C > 0$ such that*

$$\|\varphi\|_1 \leq C \|P\varphi\|_\ell \quad \text{for all } \varphi \in V \cap C^\infty(\mathbb{T}^N). \quad (2.13)$$

Proof. Suppose that (2.13) does not hold. Then for each $\ell \in \mathbb{Z}_+$ fixed we can select a sequence $\varphi_j^\ell \in V \cap C^\infty(\mathbb{T}^N)$ such that $\|\varphi_j^\ell\|_1 = 1$, $j = 1, 2, \dots$ and $\|P\varphi_j^\ell\|_\ell < \frac{1}{j}$. Thus, $P\varphi_j^\ell \rightarrow 0$ in $H^\ell(\mathbb{T}^N)$ and therefore

$$P\varphi_j^\ell \rightarrow 0 \quad \text{in } D'(\mathbb{T}^N) \quad \text{as } j \rightarrow \infty. \quad (2.14)$$

Since $\|\varphi_j^\ell\|_1 = 1$ it follows from Rellich's Lemma that there exists a subsequence, which we keep calling φ_j^ℓ , such that

$$\varphi_j^\ell \rightarrow u_0^\ell \text{ in } L^2(\mathbb{T}^N) \quad \text{as } j \rightarrow \infty.$$

Hence

$$\varphi_j^\ell \rightarrow u_0^\ell \text{ in } D'(\mathbb{T}^N) \quad \text{as } j \rightarrow \infty,$$

and

$$P\varphi_j^\ell \rightarrow Pu_0^\ell \text{ in } D'(\mathbb{T}^N) \quad \text{as } j \rightarrow \infty. \quad (2.15)$$

By (2.14) and (2.15) we have

$$Pu_0^\ell = 0. \quad (2.16)$$

Therefore

$$u_0^\ell \in \text{Ker } P. \quad (2.17)$$

We know from (2.11) that there exist $m \in \mathbb{Z}_+$ and $C > 0$ such that

$$\|\varphi\|_1 \leq C(\|P\varphi\|_m + \|\varphi\|_0), \quad \forall \varphi \in C^\infty(\mathbb{T}^N).$$

By taking $\ell = m$, we obtain

$$\|\varphi_j^m\|_1 \leq C(\|P\varphi_j^m\|_m + \|\varphi_j^m\|_0), \quad j = 1, 2, \dots \quad (2.18)$$

Then, by taking the limit, in (2.18), as $j \rightarrow \infty$ we obtain

$$1 \leq C\|u_0^m\|_0, \quad (2.19)$$

which gives $u_0^m \neq 0$. On the other hand, observe that $u_0^m \in V$ since $\varphi_j^m \in V$, $j = 1, 2, \dots$ and $\varphi_j^m \rightarrow u_0^m$ in $L^2(\mathbb{T}^N)$. This together with (2.17) give that $u_0^m = 0$, which contradicts the fact that $u_0^m \neq 0$. \square

Corollary 2.7. *If P is globally C^∞ hypoelliptic in \mathbb{T}^N then there exist $\ell \in \mathbb{Z}_+$ and $C > 0$ such that*

$$\|\varphi\|_0 \leq C\|P\varphi\|_\ell \quad \text{for all } \varphi \in V \cap C^\infty(\mathbb{T}^N). \quad (2.20)$$

3. End of the proof of Theorem 2.2

Let $u \in D'(\mathbb{T}^N)$ be such that

$$Pu = f \in G^s(\mathbb{T}^{m+n}). \quad (3.1)$$

Since P is globally C^∞ hypoelliptic in \mathbb{T}^{m+n} it follows from (3.1) that

$$u(t, x) \in C^\infty(\mathbb{T}^{m+n}). \quad (3.2)$$

Also, since $\text{Ker } P$ is a closed subset of $L^2(\mathbb{T}^{m+n})$ we have

$$L^2(\mathbb{T}^{m+n}) = \text{Ker } P \oplus V.$$

Thus, we can write

$$u(t, x) = u_0(t, x) + u_1(t, x) \quad \text{where } u_0 \in \text{Ker } P \text{ and } u_1 \in V. \quad (3.3)$$

We have $u_0 \in G^s(\mathbb{T}^{m+n})$ since by hypothesis $\text{Ker } P \subset G^s(\mathbb{T}^{m+n})$. Therefore, to prove Theorem 2.2 it suffices to analyze the function $u_1 \in V$. Since u and u_0 belong to $C^\infty(\mathbb{T}^{m+n})$ we can conclude from (3.3) that $u_1 \in V \cap C^\infty(\mathbb{T}^{m+n})$. Thus, it follows from Corollary 2.7 that there exist $C > 0$ and $\ell \in \mathbb{Z}_+$ such that

$$\|u_1\|_0 \leq C\|Pu_1\|_\ell. \quad (3.4)$$

Since the coefficients of P depend only on t we have

$$[\partial_x^\alpha, P] = 0 \quad \text{for all } \alpha \in \mathbb{Z}_+^n, \quad |\alpha| > 0. \quad (3.5)$$

As a consequence of (3.5) we have

$$\partial_x^\alpha : V \cap C^\infty(\mathbb{T}^{m+n}) \rightarrow V \cap C^\infty(\mathbb{T}^{m+n}).$$

Hence, since $u_1 \in V \cap C^\infty(\mathbb{T}^{m+n})$ we have $\partial_x^\alpha u_1 \in V \cap C^\infty(\mathbb{T}^{m+n})$. It follows from (3.4), by replacing u_1 by $\partial_x^\alpha u_1$, and (3.5) that

$$\|\partial_x^\alpha u_1\|_0 \leq C \|P(\partial_x^\alpha u_1)\|_\ell \leq C \|\partial_x^\alpha P u_1\|_\ell. \quad (3.6)$$

Now, thanks to the fact that $u_0 \in \text{Ker } P$, it follows from (3.1) and (3.6) that

$$\|\partial_x^\alpha u_1\|_0 \leq C \|\partial_x^\alpha f\|_\ell. \quad (3.7)$$

To complete the proof, we shall need the following

Lemma 3.1. *Let $N \in \mathbb{Z}_+$, $\xi \in \mathbb{Z}^n \setminus \{0\}$, and $s \geq 1$ be given. Let also $f \in G^s(\mathbb{T}^{m+n})$ be given. Then there exists a constant $C' > 0$ such that*

$$|\xi|^{N/s} |\hat{u}_1(t, \xi)| \leq C' (C' N)^N, \quad (3.8)$$

where C' is independent of N .

Proof. First of all we note that there exists $C > 0$ such that

$$|\partial_t^\alpha \partial_x^\gamma f(t, x)| \leq C^{|\alpha|+|\gamma|+1} (\alpha! \gamma!)^s$$

for all $(t, x) \in \mathbb{T}^{m+n}$ since $f \in G^s(\mathbb{T}^{m+n})$. Thus for $|\alpha, \gamma| = |\alpha| + |\gamma| \leq \ell$ and $|\beta| = M \in \mathbb{Z}_+$ we have

$$\begin{aligned} \|\partial_t^\alpha \partial_x^\gamma (\partial_x^\beta f(t, x))\|_0^2 &= \int_{\mathbb{T}^{m+n}} |\partial_t^\alpha \partial_x^{\gamma+\beta} f(t, x)|^2 dt dx \\ &\leq \int_{\mathbb{T}^{m+n}} C^{|\alpha|+|\gamma|+|\beta|+1} (\alpha!)^s (\gamma + \beta)!^s dt dx \\ &\leq (2\pi)^{m+n} C^{\ell+M+1} |\alpha|^{s|\alpha|} 2^{|\alpha|s+|\beta|s} |\gamma|^{s|\gamma|} |\beta|^{s|\beta|} \\ &\leq (2\pi)^{m+n} C^{\ell+M+1} \ell^{s\ell} 2^{\ell s} 2^{Ms} \ell^{s\ell} M^{sM} \\ &\leq C_1^{M+1} M^{sM}, \end{aligned} \quad (3.9)$$

where C_1 is independent of M .

From now on we shall use the letter C to represent a constant which may change a finite number of times.

It follows from (3.7) and Cauchy–Schwarz inequality that

$$\begin{aligned}
 |\zeta^\beta \hat{u}_1(t, \zeta)| &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ix \cdot \zeta} \zeta^\beta u_1(t, x) dx \right| \\
 &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} (-1)^{|\beta|} D_x^\beta (e^{-ix \cdot \zeta}) u_1(t, x) dx \right| \\
 &= \left| \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ix \cdot \zeta} D_x^\beta u_1(t, x) dx \right| \\
 &\leq \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |\partial_x^\beta u_1(t, x)| dx \\
 &\leq C \|\partial_x^\beta u_1(t, x)\|_0 \\
 &\leq C \|\partial_x^\beta f(t, x)\|_\ell.
 \end{aligned} \tag{3.10}$$

Let $N \in \mathbb{Z}_+$ and $M = [N/s] + 1$, where $[N/s]$ is the greatest integer that is less than or equal to N/s . Using the formula

$$(t_1 + \cdots + t_n)^M = \sum_{|\beta|=M} \frac{M!}{\beta_1! \cdots \beta_n!} t_1^{\beta_1} \cdots t_n^{\beta_n}, \quad t_1, \dots, t_n \in \mathbb{R},$$

we obtain

$$|\xi|^M \leq (|\xi_1| + \cdots + |\xi_n|)^M = \sum_{|\beta|=M} \frac{M!}{\beta!} |\xi^\beta|, \quad \xi \in \mathbb{Z}^n \setminus \{0\}.$$

By (3.10), (3.9) and the last inequality we have

$$\begin{aligned}
 |\xi|^{N/s} |\hat{u}_1(t, \xi)| &\leq |\xi|^M |\hat{u}_1(t, \xi)| \leq \sum_{|\beta|=M} \frac{M!}{\beta!} |\xi^\beta| |\hat{u}_1(t, \xi)| \\
 &= \sum_{|\beta|=M} \frac{M!}{\beta!} |\xi^\beta \hat{u}_1(t, \xi)| \\
 &\leq C \sum_{|\beta|=M} \frac{M!}{\beta!} \|\partial_x^\beta f(t, x)\|_\ell \\
 &= C \sum_{|\beta|=M} \frac{M!}{\beta!} \left[\sum_{|(\alpha, \gamma)| \leq \ell} \|\partial_t^\alpha \partial_x^\gamma (\partial_x^\beta f(t, x))\|_0^2 \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{|\beta|=M} \frac{M!}{\beta!} C_1^{M+1} M^{sM} = C n^M C_1^{M+1} M^{sM} \\
&\leq C_2^{M+1} M^{sM},
\end{aligned} \tag{3.11}$$

where C_2 is independent of N . Thus it follows from [R, p. 32] that there exists a new constant $C' > 0$ such that

$$|\xi|^{N/s} |\hat{u}_1(t, \xi)| \leq C' (C' N)^N,$$

where C' is independent of N . The proof of Lemma 3.1 is complete. \square

Now, we are going to complete the proof of Theorem 2.2.

It follows from Lemma 3.1 and a variation of [R, Lemma 1.6.2] that there exist $C > 0$ and $\varepsilon > 0$ such that

$$|\hat{u}_1(t, \xi)| \leq C e^{-\varepsilon |\xi|^{1/s}}, \quad \xi \in \mathbb{Z}^n, \quad t \in \mathbb{T}^m. \tag{3.12}$$

Thus, there exist $C > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned}
|\hat{u}_1(\tau, \xi)| &= \frac{1}{(2\pi)^m} \left| \int_{\mathbb{T}^m} e^{-it \cdot \tau} \hat{u}_1(t, \xi) dt \right| \\
&\leq C e^{-\varepsilon |\xi|^{1/s}}, \quad (\tau, \xi) \in \mathbb{Z}^{m+n}.
\end{aligned} \tag{3.13}$$

Let $(\tau_0, \xi_0) \in \mathbb{R}^{m+n}$ with $\xi_0 \neq 0$. We define $\Gamma \doteq \{(\tau, \xi) \in \mathbb{R}^{m+n} : |\tau| < C|\xi|\}$, where C is such that $(\tau_0, \xi_0) \in \Gamma$. Thus, $(0, 0) \notin \Gamma$, $(\tau_0, 0) \notin \Gamma$, and if $(\tau, \xi) \in \Gamma$ then $\xi \neq 0$. It follows from (3.13) that

$$\begin{aligned}
|\hat{u}_1(\tau, \xi)| &\leq C e^{-\varepsilon (\frac{1}{2}|\xi| + \frac{1}{2}|\xi|)^{1/s}} \\
&\leq C e^{-\varepsilon (\frac{1}{2C}|\tau| + \frac{1}{2}|\xi|)^{1/s}} \\
&\leq C e^{-\varepsilon' |(\tau, \xi)|^{1/s}}, \quad (\tau, \xi) \in \Gamma \cap \mathbb{Z}^{m+n}.
\end{aligned} \tag{3.14}$$

Now, let $\tau_0 \in \mathbb{R}^m \setminus \{0\}$. Since, by hypothesis, $(t, x, \tau_0, 0) \notin WF_s(u)$ for any $(t, x) \in \mathbb{T}^{m+n}$ it follows that there exist positive constants ε_1, C and a cone Γ_1 containing $(\tau_0, 0)$ such that

$$|\hat{u}(\tau, \xi)| \leq C e^{-\varepsilon_1 |(\tau, \xi)|^{1/s}}, \quad \forall (\tau, \xi) \in \Gamma_1 \cap \mathbb{Z}^{m+n}. \tag{3.15}$$

Therefore, (3.14) and (3.15) give that $u \in G^s(\mathbb{T}^{m+n})$. This completes the proof of Theorem 2.2. \square

Remark. Another way to prove property (\star) when the operator P has constant coefficients, is the following. Let P be an operator with constant coefficients, i.e., $P(D_y) = \sum_{|\alpha| \leq m} a_\alpha D_y^\alpha$, $a_\alpha \in \mathbb{C}$, and let us suppose that P is globally C^∞ hypoelliptic in \mathbb{T}^N . It follows from [GW, Corollary] that if $v \in C^\infty(\mathbb{T}^N)$ is such that $Pv = 0$ then there exists a positive constant C such that

$$v(y) = \sum_{|\eta| \leq C} \hat{v}(\eta) e^{iy \cdot \eta}.$$

Thus, $v \in G^s(\mathbb{T}^N)$ since $e^{iy \cdot \eta} \in G^s(\mathbb{T}^N)$. Hence $\text{Ker } P \subset G^s(\mathbb{T}^N)$. Now, if we follow the proof of Theorem 2.2 noting that inequality (3.7) becomes

$$\|\partial_y^\alpha u_1\|_0 \leq C \|\partial_y^\alpha f\|_\ell,$$

since the derivatives with respect to any variable commute with P , we then obtain $u_1 \in G^s(\mathbb{T}^N)$. Therefore, we conclude that P is globally G^s hypoelliptic in \mathbb{T}^N .

4. Applications

Here, by using Theorem 2.2 we will prove global G^s hypoellipticity for a large class of operators which are the sum of a sublaplacian and lower order terms. More precisely, we will prove the following result.

Theorem 4.1. *In the torus \mathbb{T}^{m+1+n} we consider the operator P given by*

$$P = -\Delta_t - \sum_{j=1}^m X_j^2 + X_0,$$

where $X_j = \partial_{t_j} + \sum_{k=1}^n a_{jk}(t) \partial_{x_k}$, $j = 0, \dots, m$, with $a_{jk} \in C^\omega(\mathbb{T}^{m+1})$ and real-valued. Suppose that there exists $j_0 \in \{1, \dots, m\}$ such that the operator

$$Q = -\Delta_t - \tilde{X}_{j_0}^2$$

is globally C^∞ hypoelliptic in \mathbb{T}^{m+1+n} where $\tilde{X}_{j_0} = \sum_{k=1}^n a_{j_0 k}(t) \partial_{x_k}$. Then the operator P is globally G^s hypoelliptic in \mathbb{T}^{m+1+n} .

Remark. We would like to point out that in [HP2] we have found a necessary and sufficient condition for the operator Q to be globally C^∞ hypoelliptic in \mathbb{T}^{m+1+n} .

We shall need the following lemma.

Lemma 4.2. *Let P and Q be as in Theorem 4.1. Then P is globally C^∞ hypoelliptic in \mathbb{T}^{m+1+n} and $\text{Ker } P = \mathbb{C}$.*

Proof. Let $u \in D'(\mathbb{T}^{m+1+n})$ be such that

$$Pu = f \in C^\infty(\mathbb{T}^{m+1+n}).$$

Taking partial Fourier coefficients with respect to x we obtain

$$\left(-\Delta_t - \sum_{j=1}^m Y_j^2 + Y_0 \right) \hat{u}(t, \xi) = \hat{f}(t, \xi), \quad (4.1)$$

where $Y_j = \partial_{t_j} + i \sum_{k=1}^n a_{jk}(t) \xi_k$, $j = 0, \dots, m$. Since the operator in (4.1) is elliptic in $t \in \mathbb{T}^{m+1}$ then $\hat{u}(\cdot, \xi) \in C^\infty(\mathbb{T}^{m+1})$. Thus, if we multiply (4.1) by $\bar{\hat{u}}(t, \xi)$ and integrate by parts with respect to $t \in \mathbb{T}^{m+1}$ we obtain

$$\begin{aligned} & \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 + \sum_{j=1}^m \|Y_j \hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \\ & + i \left[\text{Im} \int_{\mathbb{T}^{m+1}} (\partial_{t_0} \hat{u}(t, \xi)) \bar{\hat{u}}(t, \xi) dt + \int_{\mathbb{T}^{m+1}} \sum_{k=1}^n a_{0k}(t) \xi_k |\hat{u}(t, \xi)|^2 dt \right] \\ & = \int_{\mathbb{T}^{m+1}} \hat{f}(t, \xi) \bar{\hat{u}}(t, \xi) dt. \end{aligned}$$

Taking the real part in the last relation, we obtain

$$\begin{aligned} & \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 + \sum_{j=1}^m \|Y_j \hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \\ & = \text{Re} \int_{\mathbb{T}^{m+1}} \hat{f}(t, \xi) \bar{\hat{u}}(t, \xi) dt. \end{aligned} \quad (4.2)$$

Since the operator $Q = -\Delta_t - \tilde{X}_{j_0}^2$ is globally C^∞ hypoelliptic in \mathbb{T}^{m+1+n} it follows from [HP2, Lemma 2.1] that there exist $C > 0$, $K \geq 0$, and $\delta > 0$ such that for each

$\xi \in \mathbb{Z}^n \setminus \{0\}$ we can find an open set I_ξ such that

$$b_{j_0}(t, \xi)^2 \doteq \left(\sum_{k=1}^n a_{j_0 k}(t) \xi_k \right)^2 \geq \frac{C}{|\xi|^K}, \quad t \in I_\xi, \quad \text{vol}(I_\xi) > \delta. \quad (4.3)$$

By using the fundamental theorem of calculus we can prove that there exists a positive constant C such that

$$\|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \leq C \left(\int_{I_\xi} |\hat{u}(s, \xi)|^2 ds + \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \right). \quad (4.4)$$

Now, using (4.4)–(4.2) we obtain

$$\begin{aligned} \|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 &\leq C \left(\int_{I_\xi} |\hat{u}(s, \xi)|^2 ds + \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \right) \\ &\leq C |\xi|^K \left(\int_{\mathbb{T}^{m+1}} b_{j_0}(t, \xi)^2 |\hat{u}(t, \xi)|^2 dt + \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \right) \\ &= C |\xi|^K \left(\int_{\mathbb{T}^{m+1}} |i b_{j_0}(t, \xi) \hat{u}(t, \xi)|^2 dt + \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \right) \\ &\leq 3C |\xi|^K \int_{\mathbb{T}^{m+1}} |\partial_{t_{j_0}} \hat{u}(t, \xi) + i b_{j_0}(t, \xi) \hat{u}(t, \xi)|^2 dt \\ &\quad + 3C |\xi|^K \sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \\ &\leq C |\xi|^K \left(\sum_{j=0}^m \|\hat{u}_{t_j}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 + \sum_{j=1}^m \|Y_j \hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}^2 \right) \\ &= C |\xi|^K \operatorname{Re} \int_{\mathbb{T}^{m+1}} \hat{f}(t, \xi) \bar{\hat{u}}(t, \xi) dt \\ &\leq C |\xi|^K \|\hat{f}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})} \|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}. \end{aligned}$$

Last relation gives

$$\|\hat{u}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})} \leq C |\xi|^K \|\hat{f}(\cdot, \xi)\|_{L^2(\mathbb{T}^{m+1})}. \quad (4.5)$$

Using (4.5) and a standard microlocal analysis argument (see [HP1]), we prove that $u \in C^\infty(\mathbb{T}^{m+1+n})$. The proof that P is globally C^∞ hypoelliptic in \mathbb{T}^{m+1+n} is now complete.

Finally we will prove that $\text{Ker } P = \mathbb{C}$. Let $u \in \text{Ker } P$, i.e., $Pu = 0$. If $f \equiv 0$ then it follows from (4.2) that $\hat{u}_{t_j}(t, \xi) = 0$, $j = 0, \dots, m$ and therefore \hat{u} is independent of t , i.e., $\hat{u}(t, \xi) = \hat{u}(\xi)$. It follows from (4.5) that $\hat{u}(\xi) = 0$ for all $\xi \in \mathbb{Z}^n \setminus \{0\}$. Hence $\text{Ker } P = \mathbb{C}$. \square

Proof of Theorem 4.1. Let $u \in D'(\mathbb{T}^{m+1+n})$ be such that $Pu = f \in G^s(\mathbb{T}^{m+1+n})$. Since P is elliptic at the points in the form $(t, x, \tau, 0)$, $(t, x) \in \mathbb{T}^{m+1+n}$, $\tau \in \mathbb{R}^{m+1} \setminus \{0\}$, we have $(t, x, \tau, 0) \notin WF_s(u)$. It follows from Lemma 4.2 that the hypotheses of Theorem 2.2 are fulfilled. Hence P is globally G^s hypoelliptic in \mathbb{T}^{m+1+n} . \square

Next, we prove global G^s hypoellipticity for a class of sublaplacians, whose global analytic hypoellipticity was studied in [CH1]. More precisely, we prove the following:

Theorem 4.3. *Let*

$$X_j = \sum_{q=1}^n a_{jq}(t) \partial_{t_q} + \sum_{k=1}^m b_{jk}(t) \partial_{x_k}, \quad j = 1, \dots, v,$$

with $a_{jk}, b_{jk} \in C^\omega(\mathbb{T}^n)$ and real-valued. Assume that:

- (i) every point in \mathbb{T}^{n+m} is of finite type for X_1, \dots, X_v ;
- (ii) $\sum a_{jq}(t) \partial_{t_q}$, $j = 1, \dots, v$ span $T(\mathbb{T}^n)_t$, for every t .

Then $P = -\sum_{j=1}^v X_j^2$ is globally G^s hypoelliptic in \mathbb{T}^{n+m} .

Proof. It follows from (i) that P is globally $C^\infty(\mathbb{T}^{n+m})$ hypoelliptic in \mathbb{T}^{n+m} , since by Hörmander's Theorem [Ho1] P is locally hypoelliptic. Next, using Bony's maximum principle (see [B]) we see that the only solutions to the equation $Pu = 0$ are the constants, and therefore $\text{Ker } P \subset G^s(\mathbb{T}^{n+m})$. Also, if $u \in D'(\mathbb{T}^{n+m})$ is such that $Pu = f \in G^s(\mathbb{T}^{n+m})$, then $(t, x, \tau, 0) \notin WF_s(u)$, $\tau \neq 0$ since, by condition (ii), P is elliptic at these points (see [Ho2, Theorem 8.6.1]). Hence the assumptions of Theorem 2.2 are fulfilled and therefore P is globally G^s hypoelliptic in \mathbb{T}^{n+m} . The proof of Theorem 4.3 is complete. \square

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